A topological characterization of dual strict convexity in Asplund spaces

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The $(*)$ property generalises the property of having a **G_δ-diagonal**, that is, the diagonal $\{(x, x) \;:\; x \in X\} \subseteq X^2$ is a G_δ set.

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Definition 1.5

Let $B \subseteq X$, where X is a Banach space. Any set of the form

$$\{x \in B \ : \ f(x) > \alpha\}, \qquad f \in S_{X^*}, \alpha \in \mathbb{R},$$

is a weakly open (w-open) **slice** of B.

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The subspace (B, w) of X has **$(*)$ with slices** if we can find a $(*)$-sequence $(\mathscr{U}_j)_{j=1}^{\infty}$, such that every element $U \in \bigcup_{j=1}^{\infty} \mathscr{U}_j$ is a w-open slice of B.

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Likewise for subspaces (B, w^*) of X^*.

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(1) $\Rightarrow$ (2): assume $\|\cdot\|$ is strictly convex on X. Given $q \in (0, \infty) \cap \mathbb{Q}$, set

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As $q < \max\{\|x\|, \|y\|\}$, there exists $f \in S_{(X, \|\cdot\|)^*}$ such that $f(x) > q$ or $f(y) > q$, so $\{x, y\} \cap \bigcup \mathscr{U}_q \neq \varnothing$.

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As $q < \max\{\|x\|, \|y\|\}$, there exists $f \in S_{(X,\|\cdot\|)^*}$ such that $f(x) > q$ or $f(y) > q$, so $\{x, y\} \cap \bigcup \mathscr{U}_q \neq \varnothing$. As $\frac{1}{2}\|x + y\| < q$, we cannot have **both** $g(x) > q$ and $g(y) > q$ for some $g \in S_{(X,\|\cdot\|)^*}$.

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The implication (4) $\Rightarrow$ (1) requires the work.

The main result

Fact 1.9

Recall that a Banach space X is **Asplund** if and only if X^* has the **Krein-Milman property**: if $B \subseteq X^*$ is norm-closed, convex and bounded, then

$$B = \overline{\mathrm{conv}}^{\|\cdot\|}(\mathrm{ext}(B)).$$

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Fact 1.10

A $C(K)$ space is Asplund if and only if K is scattered.

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The proof involves a set-theoretic derivation process indexed by a tree of finite sequences of ordered pairs of natural numbers and rational numbers.